



Problem of the Week Grade 9 and 10

Formidable Fractions? Solution

Problem

For positive integers a and c , $\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18$. Determine the number of ordered pairs (a, c) that satisfy $a + 3c \leq 99$.

Solution 1

First we should simplify the fractional equation, $\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18$

Finding common denominators, $\frac{\left(\frac{2a}{2c} + \frac{ac}{2c} + \frac{2c}{2c}\right)}{\left(\frac{2c}{ac} + \frac{2a}{ac} + \frac{ac}{ac}\right)} = 18$

Simplifying, $\frac{\left(\frac{2a + ac + 2c}{2c}\right)}{\left(\frac{2c + 2a + ac}{ac}\right)} = 18$

Multiplying by the reciprocal, $\frac{(2a + ac + 2c)}{2c} \times \frac{ac}{(2c + 2a + ac)} = 18$

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to $\frac{ac}{2c} = 18$. Since $c \neq 0$, the expression further simplifies to $\frac{a}{2} = 18$ or $a = 18(2) = 36$. Substituting $a = 36$ into $a + 3c \leq 99$ we obtain $36 + 3c \leq 99$ which simplifies to $3c \leq 63$ and $c \leq 21$ follows.

But $c \geq 1$ and c is an integer so $1 \leq c \leq 21$. The value of a is 36 for each of the 21 possible values of c .

$\therefore$ there are 21 ordered pairs (a, c) that satisfy the problem.

Solution 2 is on the next page.



**Solution 2**

First we should simplify the fractional equation, $\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} = 18$

Multiply the numerator and the denominator by $2ac$: $\frac{\left(\frac{a}{c} + \frac{a}{2} + 1\right)}{\left(\frac{2}{a} + \frac{2}{c} + 1\right)} \times \frac{2ac}{2ac} = 18$

The equation simplifies to: $\frac{2a^2 + a^2c + 2ac}{4c + 4a + 2ac} = 18$

Factoring, we obtain: $\frac{a(2a + ac + 2c)}{2(2c + 2a + ac)} = 18$

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to $\frac{a}{2} = 18$ and $a = 36$ follows.

Substituting $a = 36$ into $a + 3c \leq 99$ we obtain $36 + 3c \leq 99$ which simplifies to $3c \leq 63$ and $c \leq 21$. But $c \geq 1$ and c is an integer so $1 \leq c \leq 21$. The value of a is 36 for each of the 21 possible values of c .

$\therefore$ there are 21 ordered pairs (a, c) that satisfy the problem.

