Problem of the Week
Grade 11 and 12

Pair Possibilities
Solution

Problem

Determine all possible ordered pairs of positive integers (a, b) that satisfy

$$\frac{1}{a} + \frac{2}{b} = \frac{8}{2a+b} \text{ and } 2a + 5b \leq 54.$$

Solution

Since a and b are positive integers satisfying $2a + 5b \leq 54$, we could write out all ordered pairs that satisfy this inequality and then determine which ones also satisfy the first equation. There will be a large number of possibilities to check so we need to find a way to reduce the number of possibilities. Working with the first equation:

$$\begin{aligned}
\text{Find a common denominator:} \quad \frac{b+2a}{ab} &= \frac{8}{2a+b} \\
\text{"Cross-multiply":} \quad (b+2a)(2a+b) &= 8ab \\
\text{Expand and simplify:} \quad 4a^2 + 4ab + b^2 &= 8ab \\
\text{Rearrange:} \quad 4a^2 - 4ab + b^2 &= 0 \\
\text{Factor:} \quad (2a-b)^2 &= 0
\end{aligned}$$

It follows that $2a - b = 0$ and $b = 2a$.

Each of the ordered pairs (a, b) will look like $(a, 2a)$. We substitute $2a$ for b in the inequality obtaining $2a + 5(2a) \leq 54$ or $12a \leq 54$ or $a \leq 4.5$. Since a is a positive integer, a can only take on integer values $1, 2, 3$ and 4.

The ordered pairs follow quickly: $\{(1, 2), (2, 4), (3, 6), (4, 8)\}$.

$\therefore$ the only ordered pairs of positive integers (a, b) that satisfy $\dfrac{1}{a} + \dfrac{2}{b} = \dfrac{8}{2a+b}$ and $2a + 5b \leq 54$ are $(1, 2), (2, 4), (3, 6)$, and $(4, 8)$.