**<http://www.mathematische-basteleien.de/pentominos.htm>**

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**What are Pentominoes?**

The 12 figures above ( each made of five squares ) are called pentominoes. When you arrange 5 squares so that the squares must have in common at least one side, the pentominoes are the only 12 possible shapes that result. Because the pentominoes are similar to capital letters, they have names that are letters.

**Building Rectangles**
The main problem associated with pentominoes is to combine all 12 pieces to form rectangles.  You can form four different rectangles:

6x10 , 3x20 , 4x15 and 5x12

There are many ways to form these different shaped rectangles:

 2339 solutions (6x10), 2 solutions (3x20), 368 solutions (4x15), 1010 solutions (5x12).

You can form a rectangle 5x13, if you leave blank a pentimono (5x13 = 65 = 60 + 5).

<https://isomerdesign.com/Pentomino/> - provides solutions to ‘rectangle puzzles

****

5 x 13

**Magnification Problems**
Pentominoes with triple magnification:

|  |  |  |  |
| --- | --- | --- | --- |
| ...http://www.mathematische-basteleien.de/pent08.gif... | ...http://www.mathematische-basteleien.de/pent09.gif... |  | You build a pentomino with triple magnification.  You need nine pieces. Three pieces are left.  |

|  |
| --- |
| ...http://www.mathematische-basteleien.de/pent10.gif. |

<http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A.html>

**Beginning Pentomino Problems**

Nearly all of the problems in this section require less than the full set of twelve pentominoes, which make them a little easier.

**A-1** *3 x 5 Rectangles.* The example below shows how three pentominoes can be put together to make a 3x5 rectangle.

Find another rectangle of the same size using the *N*, *P*, and *U*pentominoes. See how many of the other five possible ways of building this rectangle you can find using different combinations of pentominoes.

**A-2** *4 x 5 Rectangles*. This rectangle is formed using four pentominoes.



**(a)** Find at least two other ways of filling a rectangle of the same size using four pentominoes.
**(b)**Find a solution where a pentomino piece does not touch the outer edge of the rectangle.
**(c)** Find a solution so that the four pentominoes used touch at the same point. (This is called a *crossroads* solution.)
**(d)** Find a solution so that the rectangle can be divided into two identical shapes.

**A-3** *5 x 5 Squares.*Now we have a square that is built from five pentominoes.



**(a)** Find at least two other ways of filling a square of the same size using five pentominoes.

**(b)**For each pentomino, try to find a solution where the given piece does not touch an edge. Does this answer change if the *I* pentomino were not used in the solution?

**A-4** *Many Rectangles.*Including the first three problems, there are 14 rectangles that can be constructed that do not use the full set of pentominoes. One of the rectangles is very simple to solve, even easier than the first problem. See if you can discover all the other different rectangles that can be constructed and a solution for each one.

**Problems A-5 and A-6:***Congruent Groups.*

These problems require two (or more) pentominoes to be put together to make one total shape, and then to find the same number of pentominoes that make the same total shape as the first one. Two groups of pentominoes that can form the same shape are called *congruent groups*. In the example below, the *I* and *L* pentominoes are congruent to the *N* and *W* pentominoes because they can form the same shape.



**A-5** Find two pentominoes which will make the same total shape as the one given below for the*I* and *U*.



**A-6** Put the *U* and *Y* pentominoes together to make the same total shape as the

**(a)** *N* and *P*,
**(b)** *N* and *Z*,
**(c)***V* and *X*,
**(d)***F* and *N*,
**(e)** *P* and *T*,
**(f)** *L* and *T*, and
**(g)** *L* and *Z*.

*Note:*The total shape for each of these will be different.

**A-7** *The Duplication Problem.*Four pentominoes can be put together to make a copy of the *P* pentomino which is two times as wide and two times as high as the original piece.



Make a duplication of any pentomino other than the *P*. Two of them cannot be done. Which ones?

Notice that the solution of the *P* duplication above in fact uses its smaller counterpart in the solution. Find a solution of the *P* duplication that does not use the *P* pentomino. For every other duplication, determine if there is a solution that does use its smaller counterpart and then determine if there is a solution that does not use its smaller counterpart.

**Problems A-8 through A-11:***Simultaneous Solutions.*

A pattern is given and you need to use some of the pentominoes to cover the pattern. You then need to use some of the remaining pentominoes to cover the same pattern. This process is called finding *simultaneous solutions* of the given pattern.

**A-8**Find three simultaneous solutions to each of the 10-square patterns below.

|  |  |  |  |
| --- | --- | --- | --- |
|  **(a)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image24.gif | **(b)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image25.gif | **(c)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image26.gif |  **(d)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image27.gif |

**A-9**Find three simultaneous solutions to each of the 15-square patterns below.

|  |  |
| --- | --- |
|   **(a)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image28.gif |  **(b)**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-A%20Images/image29.gif |

**A-10***Simultaneous Rectangles.* Determine how many pairs of rectangles can be constructed simultaneously using a single set of pentominoes. As an example, problem A-1 demonstrates that two 3x5 rectangles can be constructed simultaneously.

**A-11** *Simultaneous Duplications.* Find all pairs of pentominoes such that their duplications can be constructed simultaneously.

**A-12***A Pentomino Farm.*The image below shows the full set of twelve pentominoes arranged as a ***fence*** to enclose a ***field***. The rule used to join them is that they must touch along the full edge of a square and not just at the corners. The enclosed field has an area of 43 unit squares, but the pieces have not been used very efficiently.



The problem is to find a pentomino fence enclosing the greatest possible area. You can grade your attempts by the following table:

|  |  |
| --- | --- |
| **Area** | **Grade** |
| 120 or above | A |
| 110 - 119 | B |
| 100 - 109 | C |
| 80 - 99 | D |
| under 80 | Horrible!! |

Partial Solutions for some of the previous questions

**A-1**  *3x5 Rectangles.*   The 7 possible solutions use the following groups of pieces:

{*L, V, N*}, {*T, L, Y*}, {*L, P, V*},{*N, P, U*}, {*F, P, U*}, {*P, U, Y*}, {*P, U, V*}, and

**See** <https://isomerdesign.com/Pentomino/5x3/index.html> for images of the 7 solutions

**A-2**  *4x5 Rectangles*.  (b) There are 2 solutions, one for the *P* and the *U*.  (c)  One solution uses {*L, P, V*, *Y*}.  (d)  One solution uses {*F, L, U*, *V*}.

For all 50 possible solutions to formkng 4x5 rectangles see: <https://isomerdesign.com/Pentomino/5x4/index.html>

**A-3**  *5 x 5 Squares.*(b)  Solutions exist for the *F, P, T, U, V, W*, and *Z*.  If the rectangle does not contain the*I*, then solutions only exist for the *P, U, W*, and *Z*.

**A-4***Many Rectangles.*

<https://isomerdesign.com/Pentomino/> - provides solutions to ‘rectangle puzzles

|  |  |
| --- | --- |
| **Size of Rectangle** | **Number of Solutions** |
| 1 x 5 | 1 |
| 3 x 5 | 7 |
| 2 x 10 | 2 |
| 4 x 5 | 50 |
| 5 x 5 | 107 |
| 3 x 10 | 145 |
| 5 x 6 | 541 |
| 5 x 7 | 1396 |
| 4 x 10 | 2085 |
| 5 x 8 | 3408 |
| 3 x 15 | 201 |
| 5 x 9 | 5902 |
| 5 x 10 | 6951 |
| 5 x 11 | 4103 |

**A-5** *Congruent Groups.*{*F, T*}

**A-7** *Duplications.* The number of solutions is given in parenthesis after each piece: *F* (0), *I* (2), *L* (2), *N* (7), *P* (48), *T* (1), *U* (5), *V* (0), *W* (4), *X* (0), *Y* (2), *Z* (6).  Duplications which have solutions that use its smaller counterpart*:  I, L, N, P, U*, and *Z*.  Duplications which have solutions that do not use its smaller counterpart:  *N, P, T, U, W, Y*, and *Z*.

**A-8** *10-square Simultaneous Solutions.*  (a)  The three groups are {*N*,*Y*}, {*P*,*Z*}, and {*F*,*T*}.  (b)  The three groups are {*I*,*L*}, {*N*,*V*}, and {*T*,*Y*}.  (c)  The three groups are {*L*,*N*}, {*V,Z*}, and {P,U}.  (d)  There are two solutions.  The first is {*L N*}, {*W,Z*}, and {*P,T*}.  The second is {*L,N*}, {*W,Z*}, and {*P,Y*}.

**A-9** *15-square Simultaneous Solutions.*  (a)  One solution uses {*N,V,Z*}, {*P,W,Y*}, and {*L,U,X*}.  (b)    One solution uses {*T,V,W*}, {*I,L,P*}, and {*F,U,Y*}.  There may be others.

**A-10***Simultaneous Rectangles.* The number of solutions for each pair of simultaneous rectangles are given in the table below.  Blank spaces indicate either that either the construction is impossible (in that it would require more than 12 pieces) or that the number is given elsewhere in the table.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | **1 x 5** | **3 x 5** | **2 x 10** | **4 x 5** | **5 x 5** | **3 x 10** | **5 x 6** |
| **3 x 5** | 7 | 5 |   |   |   |   |   |
| **2 x 10** | 0 | 0 | 0 |   |   |   |   |
| **4 x 5** | 36 | 29 | 0 | 28 |   |   |   |
| **5 x 5** | 35 | 25 | 0 | 60 | 12 |   |   |
| **3 x 10** | 82 | 10 | 1 | 25 | 6 | 0 |   |
| **5 x 6** | 205 | 67 | 0 | 133 | 20 | 0 | 2 |
| **5 x 7** | 398 | 84 | 0 | 22 | 1 | - | - |
| **4 x 10** | 621 | 9 | 0 | 5 | - | - | - |
| **5 x 8** | 775 | 29 | 0 | 0 | - | - | - |
| **3 x 15** | 19 | 0 | - | - | - | - | - |
| **5 x 9** | 780 | 1 | - | - | - | - | - |
| **5 x 10** | 416 | - | - | - | - | - | - |
| **5 x 11** | 112 | - | - | - | - | - | - |

**A-12***A Pentomino Farm.*The maximum possible area is 128.

More Background

The terms polyomino and pentomino were first used by Solomon Golomb in a talk to the Harvard Mathematics Club in 1953 and a year later in an article in the *American Mathematical Monthly*. They were coined by Golomb to describe a generalization of a domino. He defined a ***polyomino*** as a set of equally-sized squares, each joined together with at least one other square along an edge.

The ***order*** of a polyomino is the number of squares used to make it. An order five polyomino is called a pentomino. The first pentomino problem was actually written much earlier in 1907 by the English inventor of puzzles, Henry Ernest Dudeny, in his book The Canterbury Puzzles. The popularity of the shapes, however, is attributed mainly to Golomb from his book Polyominoes: Puzzles, Patterns, Problems, and Packings and to Martin Gardner from his monthly articles in *Scientific American*.

The simpler polyominoes - all the possible shapes composed of fewer than five connected squares-are shown below. It is assumed that two polyominoes are the same if one can be rotated (turned 90, 180, or 270 degrees) and/or reflected (flipped over) to get the second (the polyominoes are said to be ***free*** in this case).

|  |  |  |
| --- | --- | --- |
|  One Square | Two Squareshttp://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-Intro%20Images/image13.gif | Three Squareshttp://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-Intro%20Images/image14.gif |
| Four Squareshttp://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-Intro%20Images/image15.gif |

For five squares, the twelve pentominoes resemble certain letters of the alphabet, and are labeled as such.

Five Squares



The total number of squares used for each set of polyominoes is summarized below.

|  |  |  |  |
| --- | --- | --- | --- |
|  **Order** |  **Name** |  **Total Number of Shapes** |  **Total Number of Squares Needed** |
| 1 |  *Monomino* | 1 | 1 |
| 2 |  *Domino* | 1 | 2 |
| 3 |  *Tronomo* | 2 | 6 |
| 4 |  *Tetronomo* | 5 | 20 |
| 5 |  *Pentomino* | 12 | 60 |
| 6 |  *Hexomino* | 35 | 210 |
| 7 |  *Heptomino* | 108 | 756 |
| 8 |  *Octomino* | 369 | 2952 |

The values in the above table have been calculated for pieces of much larger size using a computer. However, pieces of order 6 or larger have little practical value as the basis of a dissection puzzles due to the complexity of most of the pieces and their lack of assembling together as a complete set into square or rectangular shapes.

Pentominoes have some very interesting mathematical properties providing a nearly endless array of challenging puzzles. The most natural shapes to construct with the pentominoes are squares and rectangles. However, since the total area of the twelve pieces combined is 60 squares, constructing a square would require an 8 by 8 'checkerboard' that would leave four squares left over. This leads to some interesting patterns where the four empty squares are arranged in some symmetric way about the board.

More Problems (more challenging)

<http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B.htm>

**Section B - Intermediate Pentomino Problems**

This section contains problems that require the use of all twelve pentominoes.  You will discover that even the first ten problems are  very difficult to solve.  Perseverance and patience is essential to solve any of the problems in this section.

***Problems B-1 through B-6*:***8x8 Square with Four Unit Holes.*

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **B-1**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image001.gif | **B-2**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image002.gif | **B-3**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image003.gif  | **B-4**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image004.gif | **B-5**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image005.gif | **B-6**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image006.gif  |

***Problems B-11 through B-13***:  *Three Hearts.*

|  |  |  |
| --- | --- | --- |
| **B-11**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image011.gif | **B-12**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image012.gif | **B-13**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image013.gif |

***Problems B-14 through B-16***:  *Three Crosses.*

|  |  |  |
| --- | --- | --- |
| **B-14**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image014.gif | **B-15**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image015.gif | **B-16**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image016.gif |

***Problems B-17 through B-22****:  Variations of the 88 Square with Four Unit Holes.*  More interesting solutions can be sought for problems B-1, B-4 and B-6.  Each pattern can be divided into congruent parts.  The division is indicated by the heavy line.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **B-17**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image017.gif | **B-18**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image018.gif | **B-19**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image019.gif  | **B-20**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image020.gif | **B-21**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image021.gif | **B-22**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image022.gif |

***Problems B-23 through B-28***:  *Variations of the 610 Rectangle*.  There are over two thousand solutions to the basic 6  10 rectangular pattern.  These six problems impose additional constraints to make the pattern more challenging.

**B-23**Obtain a solution to the 610 rectangle by constructing two simultaneous 65 rectangles.  (Note that this solution can also form a 512 rectangle.)

**B-24** Build the 610 rectangle such that it contains

      **(*a*)** a 35 sub-rectangle.

      **(*b*)** a 45 sub-rectangle.

     Can the sub-rectangles be constructed so that it is completely contained *within* the larger rectangle?

**B-25** The 610 rectangle is constructed from the two congruent halves shown below to the left.  The lighter shaded portion may be shifted as a unit to form the 97 rectangle with three unit holes shown below to the right.



**B-26** In a way similar to the previous problem, the darker shaded portion of the 610 rectangle may be shifted to form a 97 rectangle.

|  |  |
| --- | --- |
|  http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image024.gif | http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image025.gif |

**B-27** Build the 610 rectangle so that

**(*a*)** every pentomino touches the edge,

**(*b*)** the *I* pentomino does not touch the edge, or

      **(*c*)** five pentominoes do not touch the edge.

**B-28** Throw the twelve pentominoes randomly on the table.  Now construct the 610 rectangle without turning over any of the pieces.

***Problems B-29 through B-36***:  *Variations of Rectangles Other Than the 610.*More interesting solutions can be sought for the 512, 415, and 320 rectangles.  The rectangles below can be divided into smaller rectangles or divided into congruent halves.  The division is indicated by a heavy line.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| **B-29**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image026.gif | **B-30**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image027.gif | **B-31**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image028.gif | **B-32**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image029.gif | **B-33**http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-B_files/image030.gif |

**B-34**



**B-35** Build the 512 rectangle so that

**(*a*)**  every pentomino touches the edge., or

**(*b*)**  four pentominoes do not touch the edge.

**B-36** Build the 415 rectangle so that two pentominoes do not touch the edge.

**B-37** Without turning pieces over from positions shown, though you are allowed to rotate them, find solutions for the 610, 512, 415, and 320 rectangles.



Polyomino

<http://mathworld.wolfram.com/Polyomino.html>

Solid Pentominos

( 5 cubes instead of 5 squares )

<http://people.rit.edu/mecsma/Professional/Puzzles/Pentominoes/P-C.htm>